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Hydromechanics of lunate-tail swimming propulsion

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This paper investigates the non-uniform motion of a thin plate of finite aspect ratio, with a rounded leading edge and sharp trailing edge, executing heaving and pitching oscillations at zero mean lift. Such vertical motions characterize the horizontal lunate tails with which cetacean mammals propel themselves, and the same motions, turned through 90° to become horizontal motions of side-slip and yaw, characterize the vertical lunate tails of certain fast-swimming fishes. An oscillating vortex sheet consisting of streamwise and spanwise components is shed to trail behind the body and it is this additional feature of the streamwise component resulting from the finiteness of the plate that makes this study a generalization of the two-dimensional treatment of lunate-tail propulsion by Lighthill (1970). The forward thrust, the power required, the energy imparted to the wake and the hydromechanical propulsive efficiency are determined for this general motion as functions of the physical parameters defining the problem: namely the aspect ratio, the reduced frequency, the feathering parameter and the position of the pitching axis. The dependence of the thrust coefficient and propulsive efficiency on these physical parameters, for the complete range of variation consistent with the assumptions of the problem, has been depicted graphically.

1. Introduction

Lighthill's (1969) study of the hydromechanics of aquatic animal locomotion brings out an important concept of the hydromechanical efficiency of the animal's propulsive flexural movements, akin to the Froude efficiency of a propeller, defined as UP/E, where U is the mean forward velocity P is the mean thrust required to overcome the viscous drag and E is the mean rate at which the body movements do work against the surrounding fluid. The optimization of the hydromechanical efficiency may have been one of the most important guiding factors in the evolution of the fast-swimming aquatic animals and flying birds. This important physical parameter depends on their propulsive modes, which Lighthill (1969), following Breder (1926), divides, with a few exceptions, into two broad classes, namely the anguilliform mode of propulsion and the carangiform mode of propulsion, the former being the pure undulatory form in which the whole body participates. In the carangiform mode the amplitude of undula-

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tion can be quite small or even zero in the anterior portion, increasing posteriorly to a large value in the neighbourhood of the trailing edge. The elongated-body theory of Lighthill (1960*a*, *b*, 1970, 1971) is applicable to both the modes of propulsion and throws light on the salient peculiarities of the fish body, namely the necking of the body anterior to the caudal fin, the dimensions of the dorsal and ventral fins, etc. Lighthill (1970) suggests that all the fastest marine animals, particularly the scombroid fishes, including the tunny fishes, various unusually fast sharks and most of the cetacean mammals, have adopted essentially the carangiform mode of propulsion with their tails converging to a high-aspect-ratio crescent-moon shape through different pathways of evolution in the pursuit of high hydromechanical propulsive efficiency. The lunate tail is horizontal and moves vertically in cetacean mammals. This arrangement is the one considered in the present paper, but all the conclusions remain valid after a rotation of axes through 90° for fishes with vertical lunate tails moving horizontally.

Elongated-body theory becomes inapplicable for crescent-moon-shaped caudal fins and a start on their analysis has been made by Lighthill (1970) using a two-dimensional linearized theory which considers the movements of any vertical section with pitch angle fluctuating in phase with its heaving velocity for different pitching axes. Lighthill (1970) stresses the need for a three-dimensional theory and here an attempt is made to lay the foundations for the future investigation of Lighthill's principal suggestion: namely that a lunate shape of the caudal fin seems to be the culminating point of the process of the evolution of the fast-moving aquatic animal in the enhancement of speed and hydromechanical propulsive efficiency.

Prandtl & Betz's (1927) concept of an infinite lifting line with sinusoidally varying strength in the spanwise direction was developed by von Kármán (1935) for finite wings, in the steady case, by means of Fourier integrals involving linear superposition of infinitely long vortex elements of sinusoidal strength. Sears (1938) presents a method for finding the lift force on a finite rectangular wing, in the unsteady case, by approximating a wing of finite span by a superposition of sinusoidal aerofoils of infinite span. Although his superposition involves only a small number of terms a more accurate investigation for a rectangular wing using the full Fourier-integral form is straightforward with modern computers. However, no attempt has so far been made to calculate the horizontal forces, of thrust or drag as the case may be, acting on the swimming plate and as this investigation is of great significance in the estimation of the propulsive efficiency a detailed study is carried out, using the concepts of von Kármán & Burgers (1934) and basing the analysis on the lifting-line assumption that the local flow around each cross-section remains two-dimensional but the local angle of incidence is influenced by the whole pattern of the time-dependent streamwise and spanwise wake vorticity. These calculations, although limited to a rectangular planform, give a clear indication of the effect of aspect ratio on lunate-tail efficiency. That efficiency, as in the two-dimensional case, is found to be greatest when the pitching axis is close to the trailing edge, which may be taken as supporting Lighthill's (1970) general arguments for the high efficiency of fin shapes with trailing edges which are nearly straight.

After having obtained a theoretical basis for calculating the instantaneous lift and moment distribution, the forward thrust due to the lift force acting on the undulating surface is obtained. This supplements the suction force, acting on the rounded leading edge owing to the fast flow around it, to give the total forward thrust. Analytical expressions for the rate of working in the execution of heaving and pitching oscillations, the energy wasted and the hydromechanical propulsive efficiency are obtained in terms of the aspect ratio of the swimming plate, the aerofoil frequency parameter $\nu c/U$ (based on the radian frequency ν, forward speed U and semi-chord c of the rectangular plate), the proportional feathering parameter θ, defined by Lighthill (1969) as the ratio of the plate slope to the slope of the path traversed by the pitching axis, and the position of the pitching axis. Numerical computations are carried out for the complete range of variation of these parameters and comparison is made with the results obtained by Lighthill (1970) for the two-dimensional case, by the acceleration-potential method.

The work has all been motivated by considerations of aquatic-animal locomotion, for which the frequency parameter takes values large enough (i.e. order unity) for unsteady effects to be really important. The conclusions may also be relevant to studies of flapping flight if used in combination with a non-zero mean angle of incidence, required for weight support. In that case the thrust would be reduced by an amount equal to the induced drag associated with the required lift. This, however, is an application where unsteady effects may often be of less significance because frequency parameters are typically lower.

2. General formulation and evaluation of lift and moment

We consider the incompressible flow generated by a thin rectangular wing moving along a straight line with mean forward velocity U and at the same time executing an oscillatory waving motion of small amplitude in the transverse direction. For large Reynolds numbers R, which is the domain of interest here, the swimming motion of the wing depends primarily on the inertial effects, which can be calculated from potential theory. The viscosity of the fluid is unimportant, except in its role of determining the vorticity shed in the wake and of producing a thin boundary layer, and hence friction at the wing surface. As the wing attains forward momentum by waving motion, the propulsive force pushes the fluid backward with a net total momentum equal and opposite to that corresponding to the force, while the frictional resistance of the wing and body gives rise to forward momentum of the fluid by entraining some of the fluid surrounding the wing and body. The momentum of reaction to inertia forces is concentrated in the vortex wake owing to the small thickness and amplitude of the undulatory trailing vortex sheet; this backward jet of fluid expelled from the wing can, however, be balanced by the momentum corresponding to the viscous drag of the wing and body and when they are cruising at a constant speed, the forward and backward momenta exactly balance. This basic mechanism of swimming propulsion has been elucidated by von Kármán & Burgers (1934) and this concept is used here to investigate the hydromechanical propulsive efficiency of a finite plate executing heaving and pitching oscillations in addition to rectilinear motion.

This problem can be attacked without excessive complication when the following assumptions used in the investigation of the steady motion of finite wings are made.

(i) The flow around each section is two-dimensional but the local angle of attack is influenced by the whole pattern of the wake vorticity.

(ii) The vertical movement of any part of the wing is small, so that the wing and every point of the trail of vortices which it leaves behind can be considered to lie in the mean plane of the wing.

(iii) The total circulation about the wing at any point is such that it produces tangential flow at the trailing edge.

Let the mean position of the wing, which has span $2s$ and chord $2c$, be a strip of the x, y plane with the y axis along the span and the origin of the co-ordinate system coincident with the centre of the wing. Let the transverse displacement of the wing, from the mean position $z = 0$, be

$$z = f(y)\,[V' - i\alpha'(x - b')]\,e^{i\nu t}, \tag{1}$$

where V' and α' are real and signify the amplitude of heaving and pitching motions, respectively, and $x = b'$, $z = 0$ is the pitching axis. A 90° phase difference between the heaving and pitching motions is assumed following Lighthill (1970), as any other phase difference represented by giving an imaginary part to V' is equivalent to a change in the position of the pitching axis.

To be strictly accurate, the Fourier representation of $f(y)$ should be taken as a Fourier integral but for numerical purposes it is necessary to evaluate the integral as a sum of discrete terms; in other words, as a Fourier series. This is equivalent to considering a problem periodic in the spanwise direction: that is, the problem of the motion of a sequence of wings spaced periodically with a suitably large horizontal period. Here we take the period as $8s$, where s is the semi-span. This implies the representation of $f(y)$ as a Fourier series

$$f(y) = \sum_{n=0}^{\infty} a_n \cos \mu y \quad (-4s < y < 4s),$$

where $\mu = n\pi/4s$ and the values of a_n can be determined from the following conditions.

(i) $f(y) = 1$ for $-s < y < s$. $\quad$ (2a)

(ii) Bound vorticity vanishes outside $-s \leqslant y \leqslant s$, i.e.

$$\Gamma(y) = 0 \quad \text{outside} \quad -s \leqslant y \leqslant s. \tag{2b}$$

A spot check using a period of $16s$ indicated very small differences from the case of a period of $8s$ reported here, presumably because the interference from additional wings as far away as 8 semi-spans is already very small.

Because of the lack of uniformity of the motion of the wing, the total circulation around the wing varies and a vortex wake having a continuous distribution of vortices is left behind. The intensity of the trailing vortices can be determined from the following conditions.

(*a*) The vorticity is restricted to the wing and the wake.

(*b*) The trailing vortices do not move behind the wing, but move with the fluid; that is to say, if the fluid is stationary, they will be stationary also, if we neglect small movements perpendicular to the direction of motion of the wing.

The wake consists of vortices with streamwise and spanwise axes and the wing itself may be replaced by a bound vortex sheet comprising spanwise and chordwise vortices. The total impulse of the vortex system, following Lamb (1957, chap. 7), is

$$I = \frac{\rho}{2}\iint_S (x\gamma_y - y\gamma_x)\,dx\,dy,$$

where ρ is the density of the fluid, x and y are the co-ordinates measured in the chordwise and spanwise directions, $\gamma_y(x,y,t)$ and $\gamma_x(x,y,t)$ are the strengths per unit area of the respective components of the vortex sheet and S is the total area of the wing and the wake. The increase in the momentum of the fluid per unit time, given by Euler's formula, results in a pressure of the fluid on the wing expressed by

$$L = -dI/dt. \tag{3}$$

This lift force includes the quasi-steady force, the force due to the added mass and the force due to the presence of the vortex wake behind the wing. With U as the mean forward velocity in the negative-x direction and

$$\Gamma(y,t) = \int_{-c}^{c} \gamma_y(x,y,t)\,dx,$$

Sears (1938) simplifies the expression for L to

$$L = -\rho\frac{d}{dt}\int_{-s}^{s}\int_{-c}^{c} x\gamma_y(x,y,t)\,dx\,dy + \rho c\frac{d}{dt}\int_{-s}^{s}\Gamma(y,t)\,dy + \rho U\int_{-s}^{s}\Gamma(y,t)\,dy \tag{4}$$

by making use of the following results.

(i) The solenoidality of the vortex intensity, i.e.

$$\partial\gamma_x/\partial x + \partial\gamma_y/\partial y = 0. \tag{5}$$

(ii) The vanishing of the spanwise and streamwise vortex strength at the wing tips and the leading edge respectively, i.e.

$$\gamma_y(x,\,\pm s,t) = 0, \quad \gamma_x(-c,y,t) = 0. \tag{6}$$

(iii) The vanishing of the total circulation for every wing cross-section, i.e.

$$\Gamma(\eta,t) + \int_c^l \gamma_y(\xi,\eta,t)\,d\xi, \tag{7}$$

where ξ and η are used to denote the co-ordinates of the points in the wake and l is the distance from the centre of the wing to the end of the discontinuity surface.

(iv) The rate at which the vortices γ_x are shed at any section along the span is determined by the rate of change of the total circulation about the wing at that section, i.e.

$$d\Gamma(\eta,t)/dt + U\gamma_y(c,\eta,t) = 0. \tag{8}$$

(v) The time derivative of the integral over the wake is given by

$$\frac{d}{dt}\int_c^l \gamma(\xi,\eta,t) f(\xi,\eta,t)\,d\xi = U\int_c^l \gamma(\xi,\eta,t)\frac{\partial f}{\partial \xi}\,d\xi + U\gamma(c,\eta,t) f(c,\eta,t). \tag{9}$$

The moment of momentum, acting about the half-chord position $x = z = 0$, for the vortices distributed along the section $(-c, l)$ is given by

$$M_m = -\frac{\rho}{2}\int_{-c}^{c} x^2\gamma_y(x,y,t)\,dx - \frac{\rho}{2}\int_c^l \xi^2\gamma_y(\xi,\eta,t)\,d\xi.$$

In the case considered, the origin O moves with velocity U, therefore if we use this result for the variation of the moment of momentum we shall be taking into account not only the variation of this moment due to the variation of the forces acting on the wing, but also its variation as a result of the movement of the mid-chord with respect to which it is defined. In order to consider the variation due to the variation of the forces and their positions we find the moment with respect to some point on the axis which is fixed in space but otherwise arbitrary. Let x_0 be the co-ordinate of the instantaneous position of O with respect to the stationary point, reserving x and ξ for the co-ordinates of the points referred to the moving point O. Then $dx_0/dt = -U$. Since the elementary vortices are stationary, for each of these vortices

$$\xi + x_0 = \text{constant}, \quad \text{yielding} \quad d\xi/dt = U.$$

The moment of momentum relative to the stationary chord position will be

$$M_m = -\frac{\rho}{2}\int_{-c}^{c} \gamma(x,y,t)\,(x+x_0)^2\,dx - \frac{\rho}{2}\int_c^l (\xi+x_0)^2\,\gamma(\xi,\eta,t)\,d\xi.$$

According to Euler's formula, the moment M of the forces acting on the wing is

$$M = -\frac{dM_m}{dt} = -\frac{\partial M_m}{dt} + U\frac{\partial M_m}{\partial x_0}.$$

Substituting for M_m, putting $x_0 = 0$ and carrying out the differentiation indicated in (9) yield

$$M = \frac{\rho}{2}\frac{d}{dt}\int_{-c}^{c} \gamma_y(x,y,t)\,x^2\,dx + \frac{\rho}{2}\frac{d}{dt}\int_c^l \gamma_y(\xi,\eta,t)\,\xi^2\,d\xi$$
$$-\rho U\left\{\int_{-c}^{c} x\gamma_y(x,y,t)\,dx + \int_c^l \gamma_y(\xi,\eta,t)\,d\xi\right\}. \tag{10}$$

This expression for the moment takes account of the moment due to the quasi-steady force (Joukowski force), the moment due to the force arising from the virtual mass and the moment due to the force resulting from the presence of the vortex wake.

Relations (4) and (10) are the general expressions for the lift and moment acting on a rectangular wing and their evaluation depends essentially on the determination of the circulation induced by the wake vortex system and the simultaneous specification of the wake pattern consistent with the circulation around the wing. For wings of sufficiently large aspect ratio, the vortex strength

$\gamma_y(x,y,t)$ and its integrals may be found using Munk's (1922, 1924) two-dimensional steady aerofoil theory, viz.,

$$\begin{aligned}\Gamma(y,t) &= \int_{-c}^{c} \gamma_y(x,y,t)\,dx \\ &= -\frac{2}{\pi}\int_{-c}^{c}\left\{\int_{-c}^{c} w(x,y,t)\left(\frac{c+x}{c-x}\right)^{\frac{1}{2}}\left(\frac{c-x_1}{c+x_1}\right)^{\frac{1}{2}}\frac{dx_1}{x_1-x}\right\}dx \\ &= 2\int_{-c}^{c} w(x,y,t)\left(\frac{c+x}{c-x}\right)^{\frac{1}{2}}dx, \end{aligned} \tag{11}$$

with $$w(x,y,t) = w_0(x,y,t)+w_1(x,y,t),$$

where w_0 is the relative normal velocity produced by the motion and the angle of attack of the wing and w_1 is the relative normal velocity induced by the wake vortex system.

Similarly, by making use of

$$u-iw = \frac{1}{\pi}\left(\frac{c-x-iz}{c+x+iz}\right)^{\frac{1}{2}}\int_{-c}^{c} -w(x,y,t)\left(\frac{c+x_1}{c-x_1}\right)^{\frac{1}{2}}\frac{dx_1}{x+iz-x_1}, \tag{12}$$

where u is the x component of fluid velocity, the expressions for

$$\int_{-c}^{c} x\gamma_y(x,y,t)\,dx, \quad \int_{-c}^{c} x^2\gamma_y(x,y,t)\,dx$$

needed in the evaluation of L and M can be calculated to be

$$\int_{-c}^{c} x\gamma_y(x,y,t)\,dx = -2\int_{-c}^{c} w(x,y,t)\,(c^2-x^2)^{\frac{1}{2}}\,dx, \tag{13}$$

$$\int_{-c}^{c} x^2\gamma_y(x,y,t)\,dx = \frac{c^2}{2}\Gamma(y,t)-2\int_{-c}^{c} x(c^2-x^2)^{\frac{1}{2}}w(x,y,t)\,dx. \tag{14}$$

From (1), consider a single harmonic, i.e. an aerofoil of infinite span, the vertical velocity $w_0(x,y,t)$ at any point is given by

$$w_0(x,y,t) = a_n\,[V-i\alpha(x-b)]\,e^{i\nu t}\cos\mu y, \tag{15}$$

where $V = i\nu V'$, $\alpha = i\nu\alpha'$ and $b = b'-U/i\nu$. If

$$\Gamma(y,t) = a_n Gc V e^{i\nu t}\cos\mu y$$

is taken as the instantaneous circulation around the aerofoil the distribution of the spanwise vortex component in the wake, making use of

$$d\Gamma/dt + U\gamma_y(x,y,t) = 0,$$

is given by $$\gamma_y(\xi,\eta,t) = -a_n\frac{i\nu}{U}Gc\exp\left[i\nu\left(t-\frac{\xi-c}{U}\right)\right]\cos\mu y. \tag{16}$$

It follows from (5) that the chordwise component is

$$\gamma_x(\xi,\eta,t) = a_n\mu Gc V\exp\{i\nu[t-(\xi-c)/U]\}\sin\mu y. \tag{17}$$

Using (16) and (17) in the Biot–Savat law for the induced velocity due to a vortex system and simplifying gives

$$\begin{aligned} w_1(x,y,t) = -\frac{i\nu}{U}a_n\frac{GcV\mu}{2\pi}e^{i\nu(t+c/U)}\cos\mu y \\ \times\int_{c}^{\infty}\left\{K_1[\mu(\xi-x)]+\frac{\mu U}{i\nu}K_0[\mu(\xi-x)]\right\}e^{-i\nu\xi/U}\,d\xi, \end{aligned} \tag{18}$$

where K_0 and K_1 are modified Bessel functions of the second kind. Substituting for $w(x,y,t)$ in (11) yields

$$\Gamma(y,t) = a_n c V \cos\mu y\, e^{i\nu t}\left[2\pi + \frac{i\alpha\pi c}{V}\left(\frac{2b}{c}-1\right) - GF_1\right], \tag{19}$$

where

$$F_1 = \frac{1}{\pi}\frac{i\nu}{U}\mu e^{i\nu c/U}\int_{-c}^{c}\int_{c}^{\infty} e^{-i\nu\xi/U}\left\{K_1[\mu(\xi-x)] + \frac{\mu U}{i\nu}K_0[\mu(\xi-x)]\right\}\left(\frac{c+x}{c-x}\right)^{\frac{1}{2}} dx\, d\xi$$

$$= m\int_1^{\infty} e^{-mt}\{I_0(mt)+I_1(mt)\}\left\{\frac{m^2+\omega^2}{(m^2t^2+\omega^2)(t^2-1)^{\frac{1}{2}}} + i\,\frac{m\omega(t^2-1)^{\frac{1}{2}}}{m^2t^2+\omega^2}\right\}dt, \tag{20a}$$

with $m = \mu c$, I_0 and I_1 modified Bessel functions of the first kind and $\omega = \nu c/U$. The constant G is found, on comparing the two expressions for $\Gamma(y,t)$, to be given by

$$G = \pi\frac{2(1-\theta) - i\theta\omega(1-2\bar{\eta})}{1+F_1},$$

where $\theta = \alpha' U/\nu V'$ is the proportional feathering parameter, expressing the ratio of the plate slope to the slope of the path traversed by the pitching axis, and $\bar{\eta}$ is the non-dimensional parameter, defined by b'/c, expressing the position of the pitching axis.

From (19) and (4) in conjunction with

$$\int_{-c}^{c} x\gamma_y(x,y,t)\,dx = -2\int_{-c}^{c} w(x,y,t)\,(c^2-x^2)^{\frac{1}{2}}\,dx$$

$$= -\pi\, Vc^2\cos\mu y e^{i\nu t}\left[1 + \frac{i\alpha b}{V} - \frac{F_2 G}{\pi}\right],$$

where

$$F_2 = \frac{i\nu\mu}{\pi c U}e^{i\nu c/U}\int_{-c}^{c}\int_{c}^{\infty} e^{-i\nu\xi/U}\left\{K_1[\mu(\xi-x)] + \frac{\mu U}{i\nu}K_0[\mu(\xi-x)]\right\}(c^2-x^2)^{\frac{1}{2}}\, d\xi\, dx$$

$$= m\int_1^{\infty} e^{-mt}(mt)^{-1}I_1(mt)\left\{\frac{(m^2+\omega^2)\,t}{(m^2t^2+\omega^2)\,(t^2-1)^{\frac{1}{2}}} + i\,\frac{m\omega\,(t^2-1)^{\frac{1}{2}}}{m^2t^2+\omega^2}\right\}dt, \tag{20b}$$

the lift per unit span is given by

$$L = \bar{L}e^{i\nu t} = \pi\rho a_n c U V e^{i\nu t}[f_1(\mu) + if_2(\mu)]\cos\mu y, \tag{21}$$

where

$$f_1(\mu) = A_1 - \frac{(B_1L_1 - B_2L_2 + C_1M_1 - C_2M_2)L_1' + (B_1L_2 + B_2L_1 + C_1M_2 + C_2M_1)L_2}{L_1'^2 + L_2^2}, \tag{22a}$$

$$f_2(\mu) = A_2 - \frac{(B_1L_2 + B_2L_1 + C_1M_2 + C_2M_1)L_1' - (B_1L_2 - B_2L_2 + C_1M_1 - C_2M_2)L_2}{L_1'^2 + L_2^2}, \tag{22b}$$

$$A_1 = 2(1-\theta) + \theta\omega^2(1-3\bar{\eta}), \qquad A_2 = \omega(3-4\theta) + 2\theta\omega\bar{\eta},$$

$$B_1 = 2(1-\theta) + \theta\omega^2(1-2\bar{\eta}), \qquad B_2 = \omega\{2(1-\theta) - \theta(1-2\bar{\eta})\},$$

$$C_1 = \omega\theta(1-2\bar{\eta}), \qquad C_2 = 2(1-\theta),$$

$$L_1' = 1 + L_1,$$

L_1 and M_1, and L_2 and M_2 being the real and imaginary parts of F_1 and F_2 respectively.

The analytic expression for the moment, which is needed in the determination of the rate of working of the plate, can be determined from (10), which with the help of (14) and (9) simplifies to

$$M = -\rho \frac{d}{dt}\int_{-c}^{c} x(c^2-x^2)^{\frac{1}{2}} w(x,y,t)\,dx - \frac{\rho c^2}{4}\frac{d\Gamma}{dt} - \rho U \int_{-c}^{c} \gamma_y(x,y,t)\,x\,dx. \quad (10a)$$

The integral in the first term on the right-hand side, making use of (15) and (18), works out to be

$$\int_{-c}^{c} x(c^2-x^2)^{\frac{1}{2}} w(x,y,t)\,dx = i\frac{\pi a_n \nu\rho V c^3}{8} e^{i\nu t}\left[i\theta\omega + \frac{GF_4}{\pi}\right], \quad (14a)$$

where

$$F_4 = \frac{i8\nu}{U}\frac{\mu}{2\pi c^2} e^{i\nu c/U}\int_{-c}^{c}\int_{c}^{\infty} e^{-i\nu\xi/U}\left\{K_1[\mu(\xi-x)] + \frac{\mu U}{i\nu}K_0[\mu(\xi-x)]\right\} x(c^2-x^2)^{\frac{1}{2}}\,d\xi\,dx$$

$$= m\int_{1}^{\infty} e^{-mt}\{I_1(mt)-I_3(mt)\}\left\{\frac{(m^2+\omega^2)\,t}{(m^2t^2+\omega^2)(t^2-1)^{\frac{1}{2}}} + i\frac{m\omega(t^2-1)^{\frac{1}{2}}}{m^2t^2+\omega^2}\right\}dt. \quad (20c)$$

Using $(10a)$ in conjunction with $(14a)$, (19) and (20) the expression for M, after involved algebraic simplifications, is given by

$$M = \overline{M}\,e^{i\nu t} = \frac{i\nu\rho c^3\ V\pi a_n}{8} e^{i\nu t}\left[G_1 + iG_2 + \frac{G_3+iG_4}{L_1'^2+L_2^2}\right], \quad (23)$$

where

$$G_1 = 8\theta\overline{\eta}, \quad G_2 = \theta\omega - 8\omega^{-1}(1-\theta), \quad G_3 = G_{11}L_1' + G_{12}L_2, \quad G_4 = G_{12}L_1' - G_{11}L_2,$$

$$G_{11} = -4(1-\theta) + 2(1-\theta)(T_1 - 8\omega^{-1}M_2) + \theta\omega(1-2\overline{\eta})(T_2 + 8\omega^{-1}M_1),$$

$$G_{12} = 2\theta\omega(1-2\overline{\eta}) + 2(T_2 + 8\omega^{-1}M_1)(1-\theta) - \theta\omega(1-2\overline{\eta})(T_1 - 8\omega^{-1}M_2)$$

and T_1 and T_2 are the real and imaginary parts of F_4.

Expressions (21) and (23) for the lift and moment reduce in the two-dimensional case to Lighthill's (1970) and Wu's (1971) results, obtained by the method of acceleration potential, which adds to the confidence in this method.

The total forward thrust results from the pressure forces exerted by the fluid on the wing and the suction force on the rounded leading edge. The contribution from the pressure force comes from its action on the surface inclined backward at an angle $i\alpha' a_n e^{i\nu t}\cos\mu y$. An important contribution to the thrust comes from the suction force acting on the rounded leading edge because of the fast flow around it. The leading-edge suction arises from the singular pressure at the leading edge and its determination requires that the nonlinear terms in the expression for the pressure in the neighbourhood of the leading edge be taken into account. As the unsteady suction force presents the same problem as that for steady motion, Blasius's theorem gives the mean suction force as

$$\tfrac{1}{2}\pi\rho|A|^2,$$

where
$$A = \frac{(2c)^{\frac{1}{2}}}{\pi}\int_{-c}^{c}\frac{w(x,y,t)}{(c^2-x^2)^{\frac{1}{2}}}dx$$
$$= \frac{(2c)^{\frac{1}{2}}}{\pi} i\nu V' a_n \cos\mu y\, e^{i\nu t}\left[\pi\left(1+\frac{i\alpha b}{V}\right)-F_3 G\right],$$
with
$$F_3 = \frac{e^{i\nu c/U}}{2\pi}\frac{i\nu c\mu}{U}\int_{-c}^{c}\int_{c}^{\infty}\frac{e^{-i\nu\xi/U}}{(c^2-x^2)^{\frac{1}{2}}}\left\{K_1[\mu(\xi-x)]+\frac{\mu U}{i\nu}K_0[\mu(\xi-x)]\right\}d\xi\, dx$$
$$= \frac{m}{2}\int_{1}^{\infty}e^{-mt}I_0(mt)\left\{\frac{(m^2+\omega^2)\,t}{(m^2t^2+\omega^2)\,(t^2-1)^{\frac{1}{2}}}+i\frac{m\omega(t^2-1)^{\frac{1}{2}}}{m^2t^2+\omega^2}\right\}dt. \qquad (20d)$$

The mean forward thrust due to the suction force resulting from a single harmonic over one complete period, $P_{1,n}$ say, is given by
$$P_{1,n} = \frac{\rho c}{\pi}\nu^2 V'^2 f_3(\mu)\int_{-4s}^{4s} a_n^2\cos^2\mu y\, dy$$
$$= 2\pi\rho c s\nu^2 V'^2 a_n^2 f_3(\mu),$$
where
$$f_3(\mu) = E_1+\frac{E_2(N_1^2+N_2^2)-E_3(L_1'N_1+L_2N_2)-E_4(L_1'N_2-L_2N_1)}{L_1'^2+L_2^2},$$
$$E_1 = (1-\theta)^2+\theta^2\omega^2\bar{\eta}^2,\quad E_2 = (1-\theta)^2+\theta^2\omega^2(\tfrac{1}{2}-\bar{\eta})^2,$$
$$E_3 = 2(1-\theta)^2-\theta^2\omega^2\eta(1-2\bar{\eta}),\quad E_4 = \theta\omega(1-\theta),$$
and N_1 and N_2 are the real and imaginary parts of $2F_3$.

The mean forward thrust due to the suction force on the leading edge of the rectangular wing resulting from the superposition of all the harmonics is given by
$$P_1 = 4\pi\rho c s\nu^2 V'^2\sum_{n=0}^{\infty}a_n^2 f_3(\mu).$$

Similarly the total mean forward thrust due to the lift force is the mean resultant of $L = \bar{L}e^{i\nu t}$ given by (21) acting on the surface (1), viz. $s\mathscr{R}[\bar{L}(-i\alpha')]a_n^2$, which on substituting the value of $\bar{L}$ yields
$$P_2 = 2\pi\rho c s\nu U V'\alpha'\sum_{n=0}^{\infty}a_n^2 f_1(\mu).$$
The total mean forward thrust due to the lift and suction forces is
$$P = 2\pi\rho c s\nu^2 V'^2\sum_{n=0}^{\infty}a_n^2\{\theta f_1(\mu)+2f_3(\mu)\}. \qquad (24)$$
The forward thrust is represented by a thrust coefficient C_T, equal to the thrust per unit wing area divided by $\frac{1}{2}\rho(\nu V')^2$. Thus $C_T = P/2\rho c s\nu^2 V'^2$, which gives
$$C_T = \pi\sum_{n=0}^{\infty}a_n^2\{\theta f_1(\mu)+2f_3(\mu)\}. \qquad (25)$$

The rate of working in combined heaving and pitching equals the lift force times the rate of heaving of the centroid, plus the pitching moment about the centroid times the rate of pitching. If these two quantities are each expressed as the real parts of the complex exponentials $a e^{i\nu t}$ and $b e^{i\nu t}$, their mean product is

$$\tfrac{1}{2}\mathscr{R}[\bar{L}\nu(-\alpha' b' - iV') + \bar{M}(-\nu\alpha')]\, a_n \cos \mu y.$$

On substituting for $\bar{L}$ and $\bar{M}$, simplifying and summing over all the harmonics, the mean rate of working of the rectangular wing comes out to be

$$E = \tfrac{1}{4}\pi\rho c s U \nu^2 V'^2 \sum_{n=0}^{\infty} a_n^2 \{8f_1(\mu) + 8\theta\omega\bar{\eta} f_2(\mu) + \theta\omega^2 f_4(\mu)\}, \quad (26)$$

where

$$f_4(\mu) = G_1 + G_3/(L_1'^2 + L_2^2).$$

The hydromechanical propulsive efficiency is defined by Lighthill (1969) as UP/E, where U is the mean forward velocity, P is the mean thrust required to overcome the viscous drag the plate would sustain when maintaining a uniform velocity U and E is the mean rate at which the flexural movements do work against the surrounding water. Using this definition

$$\text{efficiency} = \frac{8 \sum_{n=0}^{\infty} a_n^2 \{\theta f_1(\mu_1) + 2f_3(\mu)\}}{\sum_{n=0}^{\infty} a_n^2 \{8f_1(\mu) + 8\theta\omega\bar{\eta} f_2(\mu) + \theta\omega^2 f_4(\mu)\}}. \quad (27)$$

3. Numerical computations and discussion of the results

The analytical expressions for the thrust coefficient and hydromechanical propulsive efficiency are functions of $f_1(\mu), f_2(\mu), f_3(\mu)$ and $f_4(\mu)$, which in turn are known in terms of the physical parameters through $L_1, L_2, M_1, M_2, T_1, T_2, N_1$ and N_2, given by the integrals appearing in ($20a$–d). These integrals possess removable singularities at the lower limit and at infinity and their evaluation can be effected by changing the variable of integration through the transformation $t = (2u^2 - 2u + 1)/u^2$. This yields

$$L_1 = 2m^{\frac{1}{2}}(m^2+\omega^2)\int_0^1 (m\alpha)^{\frac{1}{2}} e^{-m\alpha}\{I_0(m\alpha) + I_1(m\alpha)\} \frac{u^2\,du}{(m^2\alpha_1^2 + u^4\omega^2)(1+u^2/\alpha_1)^{\frac{1}{2}}},$$

$$L_2 = 2m^{\frac{3}{2}}\omega\int_0^1 (m\alpha)^{\frac{1}{2}} e^{-m\alpha}\{I_0(m\alpha) + I_1(m\alpha)\} \frac{(1-u)^2(1+u^2/\alpha_1)^{\frac{1}{2}}\,du}{m^2\alpha_1^2 + u^4\omega^2},$$

$$M_1 = \frac{2(m^2+\omega^2)}{m^{\frac{1}{2}}}\int_0^1 (m\alpha)^{\frac{1}{2}} e^{-m\alpha} I_1(m\alpha) \frac{u^4\,du}{\alpha_1(m^2\alpha_1^2 + u^4\omega^2)(1+u^2/\alpha_1)^{\frac{1}{2}}},$$

$$M_2 = 2m^{\frac{1}{2}}\omega\int_0^1 (m\alpha)^{\frac{1}{2}} e^{-m\alpha} I_1(m\alpha) \frac{u^2(1-u)^2(u^2/\alpha_1+1)^{\frac{1}{2}}\,du}{\alpha_1(m^2\alpha_1^2 + u^4\omega^2)},$$

$$N_1 = 2m^{\frac{1}{2}}(m^2+\omega^2)\int_0^1 (m\alpha)^{\frac{1}{2}} e^{-m\alpha} I_0(m\alpha) \frac{u^2 du}{(m^2\alpha_1^2 + u^4\omega^2)(1+u^2/\alpha_1)^{\frac{1}{2}}},$$

$$N_2 = 2m^{\frac{3}{2}}\omega\int_0^1 (m\alpha)^{\frac{1}{2}} e^{-m\alpha} I_0(m\alpha) \frac{(1-u)^2(1+u^2/\alpha_1)^{\frac{1}{2}}}{m^2\alpha_1^2 + u^4\omega^2}\,du,$$

$$T_1 = 2m^{\frac{1}{2}}(m^2+\omega^2)\int_0^1 (m\alpha)^{\frac{1}{2}} e^{-m\alpha}\{I_1(m\alpha) - I_3(m\alpha)\} \frac{u^2\,du}{(m^2\alpha_1^2 + u^4\omega^2)(1+u^2/\alpha_1)^{\frac{1}{2}}},$$

$$T_2 = 2m^{\frac{3}{2}}\omega\int_0^1 (m\alpha)^{\frac{1}{2}} e^{-m\alpha}\{I_1(m\alpha) - I_3(m\alpha)\} \frac{(1+u^2/\alpha_1)^{\frac{1}{2}}(1-u)^2}{m^2\alpha_1^2 + u^4\omega^2}\,du,$$

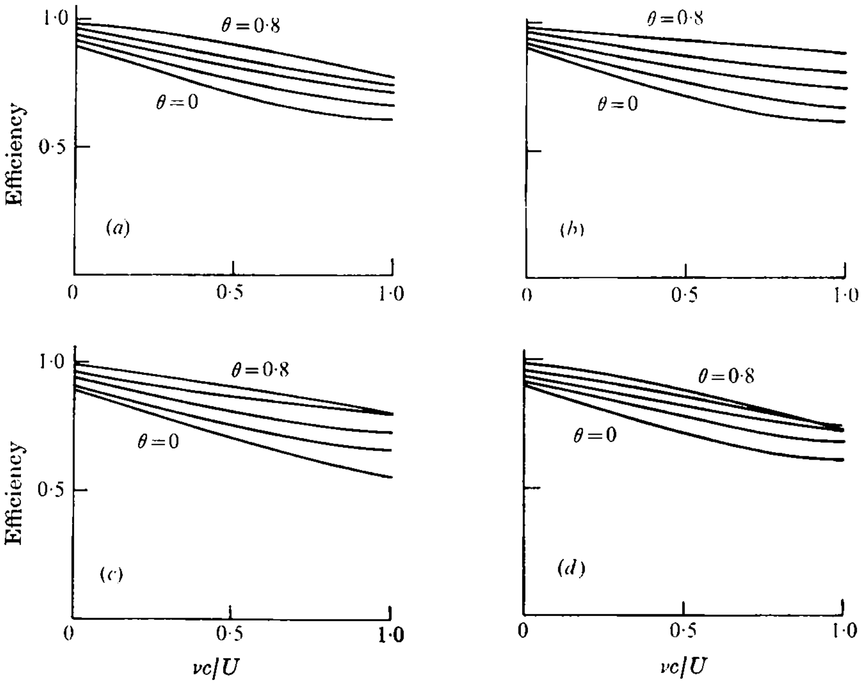

FIGURE 1. Predicted efficiency for a wing of aspect ratio 4 *vs.* $\nu c/U$ for different positions $x = b'$ of the pitching axis and different values of the feathering parameter: $\theta = \alpha' U/\nu V = 0$, 0·2, 0·4, 0·6, 0·8. (*a*) $b' = 0$ (half chord), (*b*) $b' = \frac{1}{2}c$ (three-quarter chord), (*c*) $b' = c$ (trailing edge), (*d*) $b' = \frac{3}{2}c$ (quarter chord beyond trailing edge).

where $\alpha_1 = 2u^2 - 2u + 1$ and $\alpha = \alpha_1/u^2$, and all the integrals are well behaved. To find $f_1(\mu), f_2(\mu), f_3(\mu)$ and $f_4(\mu)$ for $n = 0$ the values of L_1, M_1, L_2, M_2, N_1, N_2, T_1 and T_2 are needed. These are given by the limiting values

$$L_1 = \omega^2 \int_0^\infty e^{-x}\{I_0(x) + I_1(x)\}\frac{dx}{x^2+\omega^2}, \quad L_2 = \omega \int_0^\infty e^{-x}\{I_0(x) + I_1(x)\}\frac{x\,dx}{x^2+\omega^2},$$

$$M_1 = 1 - \int_0^\infty e^{-x} I_1(x)\frac{x\,dx}{x^2+\omega^2}, \quad M_2 = \omega\int_0^\infty e^{-x} I_1(x)\frac{dx}{x^2+\omega^2},$$

$$N_1 = \omega^2 \int_0^\infty e^{-x} I_0(x)\frac{dx}{x^2+\omega^2}, \quad N_2 = \omega\int_0^\infty e^{-x} I_0(x)\frac{x\,dx}{x^2+\omega^2},$$

$$T_1 = \omega^2 \int_0^\infty e^{-x}\{I_1(x) - I_3(x)\}\frac{dx}{x^2+\omega^2}, \quad T_2 = \omega\int_0^\infty e^{-x}\{I_1(x) - I_3(x)\}\frac{x\,dx}{x^2+\omega^2},$$

which through complicated integrations of the modified Bessel functions turn out to be

$$L_1 = -1 + \tfrac{1}{2}\pi[\{J_0(\omega) - Y_1(\omega)\}\,\omega\cos\omega + \{Y_0(\omega) + J_1(\omega)\}\,\omega\sin\omega],$$

$$L_2 = -\tfrac{1}{2}\pi[\{Y_0(\omega) + J_1(\omega)\}\,\omega\cos\omega - \{J_0(\omega) - Y_1(\omega)\}\,\omega\sin\omega],$$

$$M_1 = 1 + \tfrac{1}{2}\pi\{J_1(\omega)\cos\omega + Y_1(\omega)\sin\omega\},$$

$$M_2 = -\omega^{-1} + \tfrac{1}{2}\pi\{J_1(\omega)\sin\omega - Y_1(\omega)\cos\omega\},$$

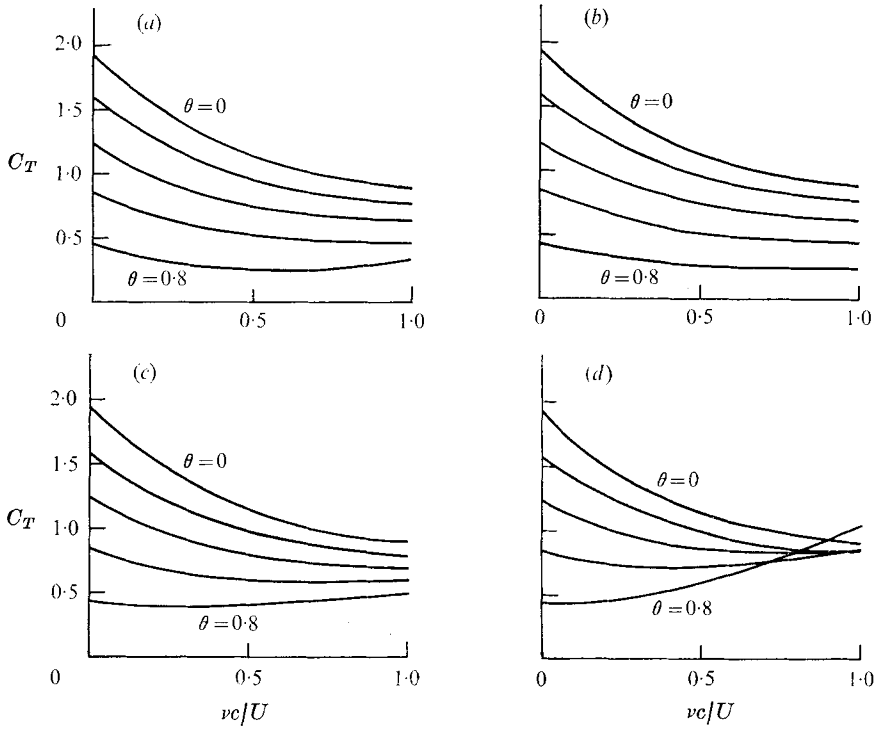

FIGURE 2. Predicted thrust coefficient for a wing of aspect ratio 4 *vs.* $\nu c/U$. θ = 0, 0·2, 0·4, 0·6, 0·8, (*a*) $b' = 0$, (*b*) $b' = \frac{1}{2}c$, (*c*) $b' = c$, (*d*) $b' = \frac{3}{2}c$.

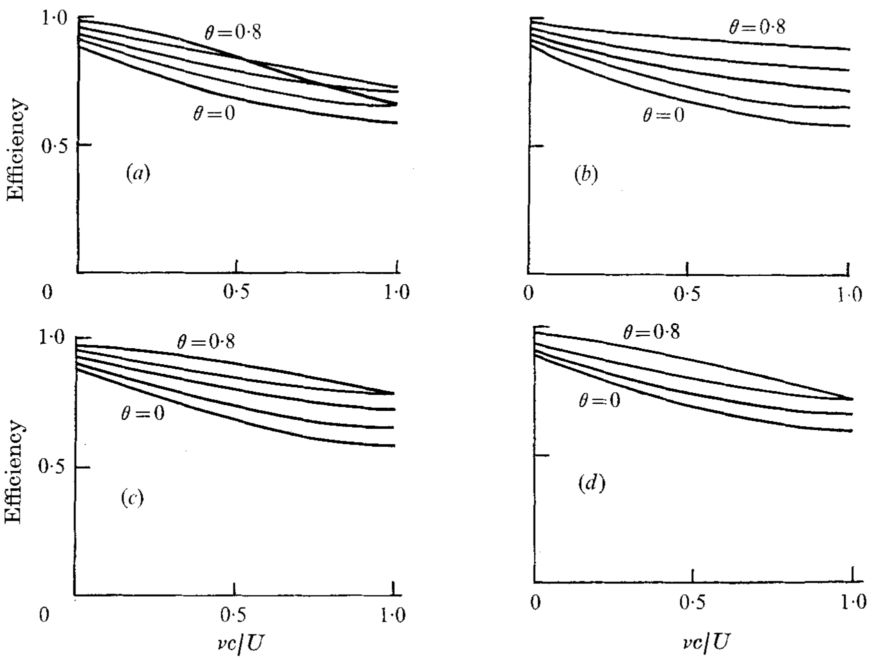

FIGURE 3. Predicted efficiency for a wing of aspect ratio 6 *vs.* $\nu c/U$. θ = 0, 0·2, 0·4, 0·6, 0·8. (*a*) $b' = 0$, (*b*) $b' = \frac{1}{2}c$, (*c*) $b' = c$, (*d*) $b' = \frac{3}{2}c$.

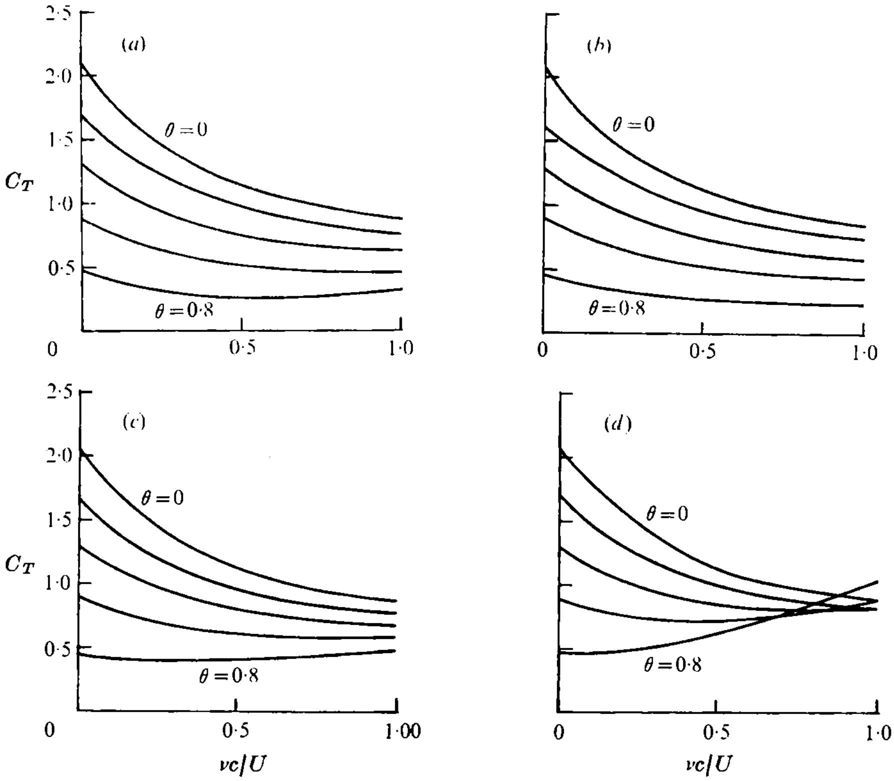

FIGURE 4. Predicted thrust coefficient foı a wing of aspect ratio 6·0 *vs.* $\nu c/U$. θ = 0, 0·2, 0·4, 0·6, 0·8. (*a*) $b' = 0$, (*b*) $b' = \frac{1}{2}c$, (*c*) $b' = c$, (*d*) $b' = \frac{3}{2}c$.

$$N_1 = \tfrac{1}{2}\pi\{Y_0(\omega)\,\omega\sin\omega + J_0(\omega)\,\omega\cos\omega\},$$

$$N_2 = -\tfrac{1}{2}\pi\{Y_0(\omega)\,\omega\cos\omega - J_0(\omega)\,\omega\sin\omega\},$$

$$T_1 = 2-\frac{8}{\omega^2}+2\pi\left[\left\{Y_0(\omega)-\frac{2}{\omega}Y_1(\omega)\right\}\cos\omega-\left\{J_0(\omega)-\frac{2}{\omega}J_1(\omega)\right\}\sin\omega\right],$$

$$T_2 = -\frac{8}{\omega}+2\pi\left[\left\{Y_0(\omega)-\frac{2}{\omega}Y_1(\omega)\right\}\sin\omega+\left\{J_0(\omega)-\frac{2}{\omega}J_1(\omega)\right\}\cos\omega\right].$$

If we seek the value of the thrust coefficient and hydromechanical efficiency for zero reduced frequency, the above expressions for $n = 0$ yield, on making use of the asymptotic expansions for the Bessel functions, $L_1 = 0$, $L_2 = 0$, $\omega M_1 = 0$, $\omega M_2 = 0$, $N_1 = 0$, $N_2 = 0$, $\omega^2 T_1 = 0$, and $\omega^2 T_2 = 0$. For $\omega = 0$ and $n \neq 0$ only L_1 and N_1 are non-zero and the expression for the hydromechanical propulsive efficiency reduces to

$$\text{efficiency} = \sum_{n=1}^{\infty} a_n^2\left\{\frac{\theta}{1+L_1}+(1-\theta)\left(1-\frac{N_1}{1+L_1}\right)^2\right\}\Big/\sum_{n=1}^{\infty}\frac{a_n^2}{1+L_1},$$

which works out to be less than unity as expected because part of the energy is wasted in the formation of the wake.

Numerical computations of the integrals involved have been carried out by

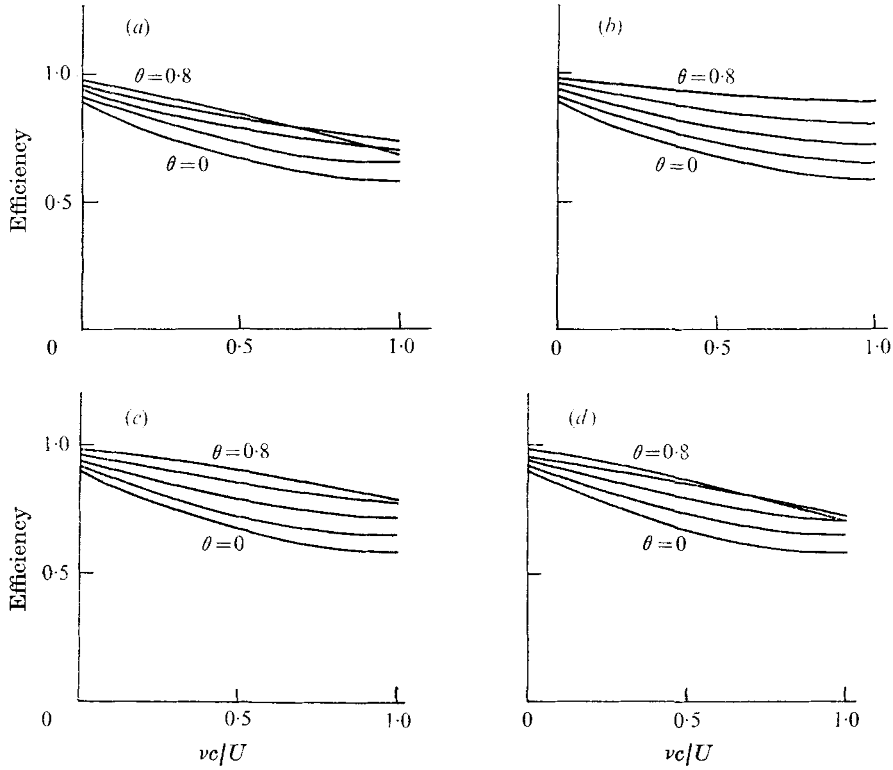

FIGURE 5. Predicted efficiency for a wing of aspect ratio 8 *vs.* $\nu c/U$. θ = 0, 0·2, 0·4, 0·6, 0·8. (*a*) $b' = 0$, (*b*) $b' = \frac{1}{2}c$, (*c*) $b' = c$, (*d*) $b' = \frac{3}{2}c$.

Romberg's extrapolation method and to obtain values of the thrust coefficient and hydromechanical propulsive efficiency correct to two decimal places twenty terms of the series occurring in (25) and (27) are needed. The coefficients of the Fourier series can be determined by satisfying (i) conditions (2*a*) for $y = \frac{1}{10}(2m-1)s$ for $m = 1, 2, \ldots, 5$ and (ii) condition (2*b*) for $y = \frac{1}{10}(2m-1)s$ for $m = 6, 7, \ldots, 20$. Condition (2*b*) becomes on using (12) and simplifying

$$\sum_{n=0}^{19} a_n \cos \mu y \left[1 - \left\{L_1 L_1' + L_2^2 + \frac{\theta\omega(1-2\bar{\eta})}{2(1-\theta)} L_2\right\} \Big/ (L_1'^2 + L_2^2)\right] = 0.$$

Thus
$$\sum_{n=0}^{19} a_n \cos \tfrac{1}{40}\pi n(2m-1) = 1 \quad \text{for} \quad m = 1, 2, \ldots, 5,$$
and
$$\sum_{n=0}^{19} a_n \cos \frac{\pi n(2m-1)}{40} \left[1 - \left\{L_1 L_1' + L_2^2 + \frac{\theta\omega(1-2\bar{\eta})}{2(1-\theta)} L_2\right\} \Big/ (L_1'^2 + L_2^2)\right] = 0$$
$$\text{for} \quad m = 6, 7, \ldots, 20$$

constitute a system of twenty linear algebraic equations, which are solved by a Gaussian method to yield the values of the coefficients a_n. Substituting these values in (25) and (27) the values of the thrust coefficient and efficiency are found for the complete range of variation of the physical parameters consistent with

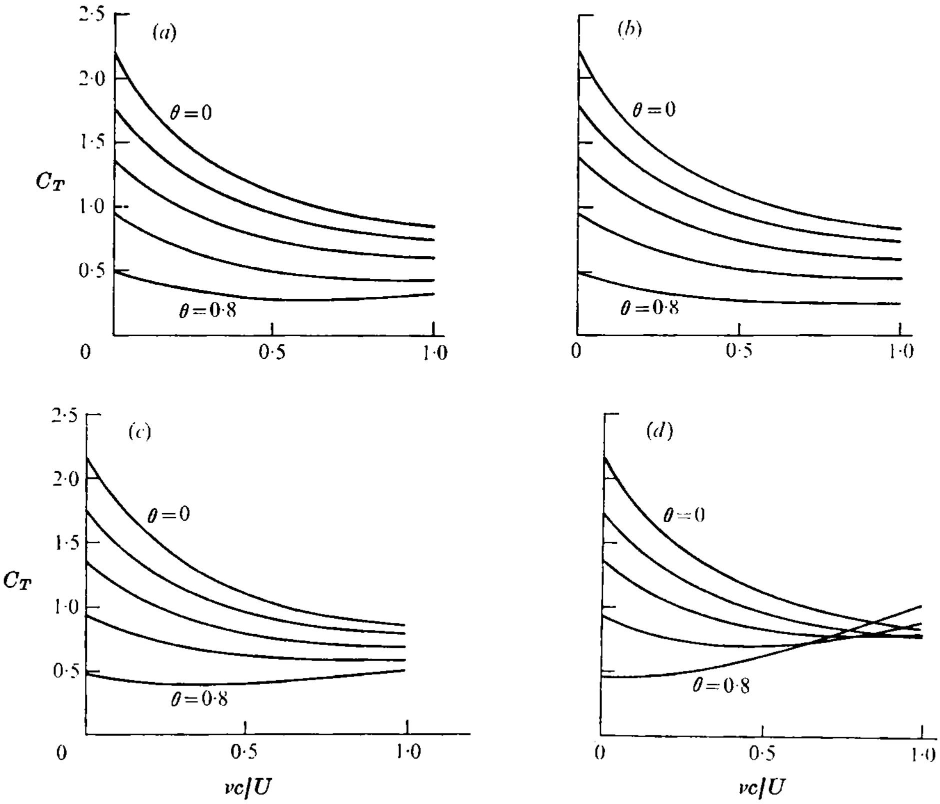

FIGURE 6. Predicted thrust coefficient for a wing of aspect ratio 8 *vs.* $\nu c/U$. θ = 0, 0·2, 0·4, 0·6, 0·8. (*a*) $b' = 0$, (*b*) $b' = \frac{1}{2}c$, (*c*) $b' = c$, (*d*) $b' = \frac{3}{2}c$.

the assumptions of the problem. Computations for different wing aspect ratios show that a decrease in the aspect ratio results in a decrease in the thrust coefficient. Sample curves for aspect ratios of 4, 6 and 8 are given (in figures 1–6) to stress the findings of the analysis. It may be pointed out that these are the aspect ratios of many of the fins of the fast-moving fishes.

The thrust coefficient decreases as the feathering parameter increases, which implies that for significant positive thrust θ must be less than one. Comparison with Lighthill's (1970) result shows that finiteness results in a considerable decrease in the forward thrust. It is noticed that, for larger values of θ which are essential for maintaining high efficiency, the thrust values are greater for positions of the pitching axis which are farther downstream and this increase is significant only for higher values of the feathering parameter and reduced frequency. This result is found to be consistent with Lighthill's (1970) observations. The increase in the thrust coefficient with the downstream shift of the pitching axis, significant for larger values of θ and $\nu c/U$, is due to a steep increase in the leading-edge suction force which also accounts for the fall in the backward-inclined lift force. This high suction force on which we are relying for optimum thrust may not be realized owing to a possible separation of the flow, which will result in a considerable decrease in the thrust coefficient. This indicates that forcing of the pitching axis downstream behind the trailing edge to achieve a high thrust coefficient is not advisable, which is consistent with Lighthill's (1970) two-dimensional theory.

The main feature of this analysis is that it gives the modified values of the thrust coefficient and propulsive efficiency for a wide variety of the physical parameters, taking into account the streamwise wake vorticity, and also confirms that a reduction in the aspect ratio of the wing results in a decrease in the thrust coefficient and the hydromechanical efficiency.

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